

Some remarks about solenoids, 3

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Abstract

A basic class of constructions is considered, in connection with bilipschitz mappings in particular.

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1 A basic situation

Let X be a (nonempty) Hausdorff topological space, and suppose that ϕ is a homeomorphism from X onto itself. Thus $X \times [0, 1]$ is also a Hausdorff space with respect to the product topology, using the standard topology on the unit interval $[0, 1]$. Let \sim_1 be the equivalence relation on $X \times [0, 1]$ in which every element of $X \times [0, 1]$ is equivalent to itself, and otherwise

$$(1.1) \quad (x, 0) \sim_1 (\phi(x), 1)$$

for every $x \in X$. This leads to a quotient space

$$(1.2) \quad Y_1 = (X \times [0, 1]) / \sim_1,$$

where the two ends $X \times \{0\}$ and $X \times \{1\}$ of $X \times [0, 1]$ are glued together using ϕ . Let q_1 be the corresponding quotient mapping from $X \times [0, 1]$ onto Y_1 , so that

$$(1.3) \quad q_1((x, 0)) = q_1((\phi(x), 1))$$

for every $x \in X$, and otherwise q_1 is one-to-one. The quotient topology on Y_1 is defined as usual by saying that $U \subseteq Y_1$ is an open set in Y_1 if and only if $q_1^{-1}(U)$ is an open set in $X \times [0, 1]$. In particular, q_1 is automatically continuous with respect to the quotient topology on Y_1 , and it is easy to see that Y_1 is also a Hausdorff space under these conditions. If X is compact, then $X \times \mathbf{R}$ is compact too, and hence Y_1 is compact with respect to the quotient topology.

Let \sim be the equivalence relation on $[0, 1]$ in which every element of $[0, 1]$ is equivalent to itself, and 0 is equivalent to 1. Thus the quotient topological space

$$(1.4) \quad [0, 1] / \sim$$

is obtained by gluing the ends of $[0, 1]$ together, and is homeomorphic to the unit circle \mathbf{S}^1 with the standard topology. The obvious coordinate projection from $X \times [0, 1]$ onto $[0, 1]$ leads to a continuous mapping from Y_1 onto (1.4), whose fibers are homeomorphic to X . If ϕ is the identity mapping on X , then Y_1 is homeomorphic to the product of X and (1.4) in a simple way.

Alternatively, let Φ be the mapping from $X \times \mathbf{R}$ into itself defined by

$$(1.5) \quad \Phi((x, t)) = (\phi(x), t + 1)$$

for every $x \in X$ and $t \in \mathbf{R}$. Thus Φ is a homeomorphism from $X \times \mathbf{R}$ onto itself. If n is a positive integer, then

$$(1.6) \quad \Phi^n((x, t)) = (\phi^n(x), t + n)$$

for every $x \in X$ and $t \in \mathbf{R}$, where ϕ^n and Φ^n are the n -fold compositions of these mappings on the corresponding spaces. This also works for $n = 0$, where the n -fold composition is interpreted as being the identity mapping on the appropriate space, and when n is a negative integer, for which the n -fold composition is considered to be the $(-n)$ -fold composition of the inverse mapping.

The collection of mappings Φ^n with $n \in \mathbf{Z}$ is a group of homeomorphisms on $X \times \mathbf{R}$. This leads to an equivalence relation \sim_2 on $X \times \mathbf{R}$, where

$$(1.7) \quad (x, t) \sim_2 (x', t')$$

for some $x, x' \in X$ and $t, t' \in \mathbf{R}$ if and only if there is an integer n such that

$$(1.8) \quad \Phi^n((x, t)) = (x', t').$$

Let q_2 be the quotient mapping from $X \times \mathbf{R}$ onto the quotient space

$$(1.9) \quad Y_2 = (X \times \mathbf{R}) / \sim_2.$$

As before, the quotient topology on Y_2 is defined by saying that $U \subseteq Y_2$ is an open set if and only if $q_2^{-1}(U)$ is an open set in $X \times \mathbf{R}$, so that the quotient mapping q_2 is automatically continuous.

Let us consider the restriction of q_2 to $X \times [0, 1] \subseteq X \times \mathbf{R}$. By construction, if $x, x' \in X$ and $t, t' \in [0, 1]$, then

$$(1.10) \quad q_1((x, t)) = q_1((x', t'))$$

in Y_1 if and only if

$$(1.11) \quad q_2((x, t)) = q_2((x', t'))$$

in Y_2 . This leads to a mapping from Y_1 into Y_2 , which is easily seen to be a homeomorphism from Y_1 onto Y_2 . An advantage of Y_2 is that q_2 is a local homeomorphism from $X \times \mathbf{R}$ onto Y_2 . Although q_1 is a local homeomorphism around points $(x, t) \in X \times (0, 1)$, this does not work when $t = 0$ or 1 .

Of course, the real line \mathbf{R} is a commutative topological group with respect to addition, which contains \mathbf{Z} as a discrete subgroup. The quotient \mathbf{R}/\mathbf{Z} is also a commutative topological group with respect to the quotient topology and group operation, which is isomorphic as a topological group to the multiplicative group of complex numbers with modulus 1. The obvious coordinate projection from $X \times \mathbf{R}$ onto \mathbf{R} leads to a continuous mapping from Y_2 onto \mathbf{R}/\mathbf{Z} . This mapping corresponds exactly to the continuous mapping from Y_1 onto (1.4) discussed earlier, using the identification between Y_1 and Y_2 described in the previous paragraph. This also uses the analogous identification between (1.4) and \mathbf{R}/\mathbf{Z} .

If $r \in \mathbf{R}$, then

$$(1.12) \quad (x, t) \mapsto (x, t + r)$$

defines a homeomorphism from $X \times \mathbf{R}$ onto itself that preserves the equivalence relation \sim_2 . This leads to a homeomorphism A_r from Y_2 onto itself, where

$$(1.13) \quad A_r(q_2((x, t))) = q_2((x, t + r))$$

for every $x \in X$ and $t \in \mathbf{R}$. This is actually a group of homeomorphisms from Y_2 onto itself, in the sense that

$$(1.14) \quad A_r \circ A_{r'} = A_{r+r'}$$

for each $r, r' \in \mathbf{R}$, because of the analogous property of (1.12) on $X \times \mathbf{R}$. Note that

$$(1.15) \quad A_n(q_2((\phi^n(x), t))) = q_2((x, t))$$

for each $x \in X$, $t \in \mathbf{R}$, and $n \in \mathbf{Z}$.

It is easy to see that

$$(1.16) \quad \psi_t(x) = q_2((x, t))$$

defines a homeomorphism from X onto $q_2(X \times \{t\})$ for each $t \in \mathbf{R}$, where $q_2(X \times \{t\})$ is equipped with the topology induced by the one on Y_2 . The sets $q_2(X \times \{t\})$ with $t \in \mathbf{R}$ are the fibers of the natural projection from Y_2 onto \mathbf{R}/\mathbf{Z} , which satisfy the periodicity condition

$$(1.17) \quad q_2(X \times \{t + 1\}) = q_2(X \times \{t\})$$

for each $t \in \mathbf{R}$, by construction. More precisely,

$$(1.18) \quad \psi_{t+1}(\phi(x)) = q_2((\phi(x), t+1)) = q_2(x, t) = \psi_t(x)$$

for every $x \in X$ and $t \in \mathbf{R}$, which implies that $\psi_{t+1}(X) = \psi_t(X)$. Similarly,

$$(1.19) \quad A_r(\psi_t(x)) = \psi_{r+t}(x)$$

for every $x \in X$ and $r, t \in \mathbf{R}$, and A_r maps $q_2(X \times \{t\})$ onto $q_2(X \times \{r+t\})$ for each $r, t \in \mathbf{R}$.

2 Connectedness

Let us continue with the notation and hypotheses in the previous section. Let $x \in X$ be given, and consider

$$(2.1) \quad q_2(\{x\} \times \mathbf{R}) = \{A_r((x, 0)) : r \in \mathbf{R}\}.$$

If $\phi^n(x) \neq x$ for every positive integer n , then it is easy to see that the restriction of q_2 to $\{x\} \times \mathbf{R}$ is a one-to-one mapping into Y_2 . Otherwise, if $\phi^n(x) = x$ for some $x \in \mathbf{Z}_+$, then $q_2((x, t))$ is periodic in t , with period n . Note that (2.1) is a connected set in Y_2 for each $x \in X$, because the real line is connected. If X is totally disconnected, then the subsets of $X \times \mathbf{R}$ of the form $\{x\} \times \mathbf{R}$ for some $x \in X$ are the pathwise-connected components of $X \times \mathbf{R}$. In this case, the subsets of Y_2 of the form (2.1) for some $x \in X$ are the pathwise-connected components of Y_2 . Of course, if X is connected, then $X \times \mathbf{R}$ is connected, and hence $Y_2 = q_2(X \times \mathbf{R})$ is connected too.

If $E \subseteq Y_2$ is both open and closed, then it follows that for each $x \in X$, (2.1) is either contained in E or in $Y_2 \setminus E$. Equivalently, this means that

$$(2.2) \quad A_r(E) = E$$

for each $r \in \mathbf{R}$, so that E is invariant under the flow on Y_2 defined by A_r . Put

$$(2.3) \quad E_0 = \{x \in X : q_2(\{x\} \times \mathbf{R}) \subseteq E\},$$

and observe that $\phi(E_0) = E_0$, since

$$(2.4) \quad q_2(\{\phi(x)\} \times \mathbf{R}) = q_2(\{x\} \times \mathbf{R})$$

for each $x \in X$. Alternatively,

$$(2.5) \quad E_0 \times \mathbf{R} = q_2^{-1}(E),$$

which is automatically invariant under Φ , and

$$(2.6) \quad E_0 = \{x \in X : \psi_t(x) \in E\}$$

for each $t \in \mathbf{R}$. This implies that E_0 is both open and closed in X . Note too that $E_0 \neq \emptyset$ when $E \neq \emptyset$, and that $E_0 \neq X$ when $E \neq Y_2$. It follows that

if Y_2 is not connected, then there is an open and closed set $E_0 \subseteq X$ such that $E_0 \neq \emptyset, X$ and $\phi(E_0) = E_0$.

Conversely, suppose that $E_0 \subseteq X$ is both open and closed in X , and that $\phi(E_0) = E_0$. This implies that

$$(2.7) \quad \Phi(E_0 \times \mathbf{R}) = E_0 \times \mathbf{R},$$

and we put

$$(2.8) \quad E = q_2(E_0 \times \mathbf{R}),$$

which is automatically invariant under A_r for each $r \in \mathbf{R}$. Observe that

$$(2.9) \quad Y_2 \setminus E = q_2((X \setminus E_0) \times \mathbf{R}),$$

because $E_0 \times \mathbf{R}$ is invariant under Φ , and hence that E is both open and closed in Y_2 . If $E_0 \neq \emptyset, X$, then $E \neq \emptyset, Y_2$, and thus Y_2 is not connected. This shows that Y_2 is connected if and only if there is no set $E_0 \subseteq X$ such that E_0 is both open and closed in X , $E_0 \neq \emptyset, X$, and $\phi(E_0) = E_0$.

If $x_0 \in X$ and the orbit

$$(2.10) \quad \{\phi^n(x_0) : n \in \mathbf{Z}\}$$

of x_0 under ϕ is dense in X , then one can check that $q_2(\{x_0\} \times \mathbf{R})$ is dense in Y_2 . This implies that Y_2 is connected, since the closure of a connected set is connected. Alternatively, if $E_0 \subseteq X$ satisfies $\phi(E_0) = E_0$, then the orbit of every element of X under ϕ is either contained in E_0 or in $X \setminus E_0$. If E_0 is also both open and closed in X , then E_0 and $X \setminus E_0$ are both closed sets in X , and hence the closure of the orbit of every element of X under ϕ is contained in E_0 or $X \setminus E_0$. If additionally $E_0 \neq \emptyset, X$, so that E_0 and $X \setminus E_0$ are both proper subsets of X , then it follows that the closure of the orbit of any element of X under ϕ is proper subset of X as well.

3 Topological groups

Let G be a topological group, and let h be an element of G . Thus

$$(3.1) \quad \phi(x) = xh$$

defines a homeomorphism from G onto itself, and

$$(3.2) \quad \phi^n(x) = xh^n$$

for each $n \in \mathbf{Z}$. Note that $G \times \mathbf{R}$ is also a topological group, where the group operations are defined coordinatewise, and using the product topology. Let H be the subgroup of $G \times \mathbf{R}$ consisting of (h^n, n) for each integer n , which is a discrete subgroup of $G \times \mathbf{R}$. Thus the quotient space $(G \times \mathbf{R})/H$ of left cosets of H in $G \times \mathbf{R}$ can be defined in the usual way, with the quotient topology on $(G \times \mathbf{R})/H$ associated to the product topology on $G \times \mathbf{R}$. The quotient space

$(G \times \mathbf{R})/H$ corresponds exactly to the space Y_2 in Section 1, and the natural quotient mapping from $G \times \mathbf{R}$ onto $(G \times \mathbf{R})/H$ corresponds to the mapping q_2 in Section 1. If the subgroup of G generated by h is normal, then H is a normal subgroup in $G \times \mathbf{R}$, and $(G \times \mathbf{R})/H$ is a topological group as well. Otherwise, $G \times \mathbf{R}$ acts on the quotient space $(G \times \mathbf{R})/H$ by left translations. If h is the identity element in G , then $(G \times \mathbf{R})/H$ reduces to $G \times (\mathbf{R}/\mathbf{Z})$.

Of course, the subgroup of G generated by h is abelian, and hence its closure in G is abelian. In particular, if the subgroup of G generated by h is dense in G , then G is abelian. This would also imply that $(G \times \mathbf{R})/H$ is connected, as in the previous section. If $G = \mathbf{Z}$ as a discrete group with respect to addition and $h = 1$, then it is easy to see that $(G \times \mathbf{R})/H$ is isomorphic as a topological group to \mathbf{R} . Alternatively, let p be a prime number, and let \mathbf{Z}_p be the group of p -adic integers. This is a compact totally disconnected commutative topological group with respect to addition, which contains \mathbf{Z} as a dense subgroup. If we take $h = 1$ as an element of \mathbf{Z}_p , then the corresponding quotient $(G \times \mathbf{R})/H$ is a compact commutative topological group which is connected but not locally connected.

If there is a countable local base for the topology of G at the identity element, then a famous theorem states that there is a metric on G that determines the same topology and which is invariant under right translations. We shall look at isometric mappings more broadly in the next section.

4 Isometries

Let us return now to the setting of Section 1. Suppose in addition that the topology on X is determined by a metric $d(x, y)$, and that ϕ is an isometric mapping from X onto itself, so that

$$(4.1) \quad d(\phi(x), \phi(y)) = d(x, y)$$

for every $x, x' \in X$. Put

$$(4.2) \quad \rho((x, r), (y, t)) = \max(d(x, y), |r - t|)$$

for each $x, y \in X$ and $r, t \in \mathbf{R}$, which defines a metric on $X \times \mathbf{R}$ for which the corresponding topology is the product topology. Thus

$$(4.3) \quad \rho(\Phi((x, r)), \Phi((y, t))) = \rho((\phi(x), r + 1), (\phi(y), t + 1)) = \rho((x, r), (y, t))$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$, where Φ is the mapping from $X \times \mathbf{R}$ onto itself defined in Section 1.

The corresponding quotient metric on Y_2 is defined by

$$(4.4) \quad \begin{aligned} D(q_2((x, r)), q_2((y, t))) \\ = \inf\{\rho((x', r'), (y', t')) : x', y' \in X, r', t' \in \mathbf{R}, \\ q_2((x', r')) = q_2((x, r)), q_2((y', t')) = q_2((y, t))\} \end{aligned}$$

for each $x, y \in X$ and $r, t \in \mathbf{R}$. Equivalently,

$$\begin{aligned}
(4.5) \quad & D(q_2((x, r)), q_2((y, t))) \\
&= \inf\{\rho((x', r'), (y, t)) : x' \in X, r' \in \mathbf{R}, q_2((x', r')) = q_2((x, r))\} \\
&= \inf\{\rho((x, r), (y', t')) : y' \in X, t' \in \mathbf{R}, q_2((y', t')) = q_2((y, t))\},
\end{aligned}$$

because Φ is an isometry on $X \times \mathbf{R}$ with respect to $\rho(\cdot, \cdot)$. If $x, x', y, z, z' \in X$ and $r, r', t, u, u' \in \mathbf{R}$ satisfy $q_2((x', r')) = q_2((x, r))$ and $q_2((z', u')) = q_2((z, u))$, then

$$\begin{aligned}
(4.6) \quad & D(q_2((x, r)), q_2((z, u))) \leq \rho((x', r'), (z', u')) \\
&\leq \rho((x', r'), (y, t)) + \rho((y, t), (z', u')),
\end{aligned}$$

by the triangle inequality for $\rho(\cdot, \cdot)$. Taking the infimum over (x', r') and (z', u') , we get that

$$\begin{aligned}
(4.7) \quad & D(q_2((x, r)), q_2((z, u))) \\
&\leq D(q_2((x, r)), q_2((y, t))) + D(q_2((y, t)), q_2((z, u))).
\end{aligned}$$

Thus $D(\cdot, \cdot)$ satisfies the triangle inequality on Y_2 , and it is easy to see that $D(\cdot, \cdot)$ is a metric on Y_2 that defines the same topology on Y_2 as before.

By construction,

$$(4.8) \quad D(q_2((x, r)), q_2((y, t))) \leq \rho((x, r), (y, t))$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$. Suppose that $r, t \in \mathbf{R}$ satisfy

$$(4.9) \quad |r - t| \leq 1/2,$$

so that

$$(4.10) \quad |r - t'| \geq 1/2$$

for every $t' \in \mathbf{R}$ such that $t' - t \in \mathbf{Z}$ and $t' \neq t$. This implies that

$$(4.11) \quad \rho((x, r), (y', t')) \geq |r - t'| \geq 1/2$$

for every $x, y, y' \in X$ and $t' \in \mathbf{R}$ such that $q_2((y', t')) = q_2((y, t))$ and $(y', t') \neq (y, t)$, so that

$$(4.12) \quad D(q_2((x, r)), q_2((y, t))) \geq \min(\rho((x, r), (y, t)), 1/2)$$

by (4.5). In particular, if

$$(4.13) \quad d(x, y) \leq 1/2,$$

then $\rho((x, r), (y, t)) \leq 1/2$, and hence

$$(4.14) \quad D(q_2((x, r)), q_2((y, t))) = \rho((x, r), (y, t)),$$

by (4.8) and (4.12). Similarly, if

$$(4.15) \quad d(x, y) \leq k$$

for some $k \geq 1/2$, then $\rho((x, r), (y, t)) \leq k$, and we get that

$$(4.16) \quad \rho((x, r), (y, t)) \leq 2k D(q_2((x, r)), q_2((y, t))).$$

If X is bounded with respect to $d(x, y)$, then (4.15) holds for some $k \geq 1/2$ and every $x, y \in X$. This implies that (4.16) holds for every $x, y \in X$ and $r, t \in \mathbf{R}$ that satisfy (4.9). Otherwise, for any positive real number k ,

$$(4.17) \quad d_1(x, y) = \min(d(x, y), k)$$

defines a metric on X which is topologically equivalent to $d(x, y)$. Of course, if ϕ is an isometry on X with respect to $d(x, y)$, then ϕ is an isometry on X with respect to $d_1(x, y)$ as well.

Suppose now that ϕ is not necessarily an isometry on X with respect to $d(x, y)$, but that the collection of iterates ϕ^n with $n \in \mathbf{Z}$ is equicontinuous at every point in X with respect to $d(x, y)$. This means that for each $x \in X$ and $\epsilon > 0$ there is a $\delta(x, \epsilon) > 0$ such that

$$(4.18) \quad d(\phi^n(x), \phi^n(y)) \leq \epsilon$$

for every $n \in \mathbf{Z}$ and $y \in X$ such that $d(x, y) < \delta(x, \epsilon)$. We may as well ask that X be bounded with respect to $d(x, y)$ too, since otherwise we can replace $d(x, y)$ with (4.17) for some $k > 0$, and still have the same equicontinuity condition. If we put

$$(4.19) \quad \tilde{d}(x, y) = \sup_{n \in \mathbf{Z}} d(\phi^n(x), \phi^n(y)),$$

then $\tilde{d}(x, y)$ is a metric on X , and

$$(4.20) \quad d(x, y) \leq \tilde{d}(x, y)$$

for every $x, y \in X$, since we can take $n = 0$ in (4.19). We also have that

$$(4.21) \quad \tilde{d}(x, y) \leq \epsilon$$

for every $x, y \in X$ such that $d(x, y) < \delta(x, \epsilon)$, by (4.18), and hence that $\tilde{d}(x, y)$ and $d(x, y)$ determine the same topology on X . By construction,

$$(4.22) \quad \tilde{d}(\phi(x), \phi(y)) = \tilde{d}(x, y)$$

for every $x, y \in X$, so that ϕ is an isometry on X with respect to \tilde{d} . This is a bit nicer when the collection of iterates ϕ^n with $n \in \mathbf{Z}$ is uniformly equicontinuous on X , in the sense that one can take $\delta(x, \epsilon) = \delta(\epsilon)$ independent of $x \in X$ for each $\epsilon > 0$. In this case, the identity mapping on X is uniformly continuous as a mapping from X equipped with $d(x, y)$ onto X equipped with $\tilde{d}(x, y)$.

Suppose that X is bounded with respect to $d(x, y)$, and let $C(X, X)$ be the space of continuous mappings from X onto itself. Thus the supremum metric

$$(4.23) \quad \sigma(f, g) = \sup_{x \in X} d(f(x), g(x))$$

is defined for each $f, g \in C(X, X)$, and it is easy to see that the group $\mathcal{I}(X)$ of isometric mappings from X onto itself is a topological group with respect to the topology determined by the restriction of $\sigma(f, g)$ to $\mathcal{I}(X)$. If X is compact, then one can check that $\mathcal{I}(X)$ is compact with respect to the supremum metric, using the Arzela–Ascoli theorem.

It is easy to see that the distance

$$(4.24) \quad \text{dist}(a, \mathbf{Z}) = \min_{n \in \mathbf{Z}} |a - n|$$

from $a \in \mathbf{R}$ to \mathbf{Z} satisfies

$$(4.25) \quad \text{dist}(a + b, \mathbf{Z}) \leq \text{dist}(a, \mathbf{Z}) + \text{dist}(b, \mathbf{Z})$$

for every $a, b \in \mathbf{R}$. Note that $\text{dist}(r - t, \mathbf{Z})$ is the same as the distance between the images of $r, t \in \mathbf{R}$ in \mathbf{R}/\mathbf{Z} under the natural quotient mapping from \mathbf{R} onto \mathbf{R}/\mathbf{Z} , with respect to the quotient metric on \mathbf{R}/\mathbf{Z} associated to the standard metric on \mathbf{R} . Of course,

$$(4.26) \quad \rho((x, r), (y, t)) \geq |r - t|$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$, by construction. It follows that

$$(4.27) \quad D(q_2((x, r)), q_2((y, t))) \geq \text{dist}(r - t, \mathbf{Z})$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$, by the definition of $D(q_2((x, r)), q_2((y, t)))$.

5 Bilipschitz mappings

Let us go back to the setting of Section 1, and suppose again that the topology on X is determined by a metric $d(x, y)$. Instead of asking that ϕ be an isometry on X , let us suppose that ϕ is bilipschitz, so that

$$(5.1) \quad C^{-1} d(x, y) \leq d(\phi(x), \phi(y)) \leq C d(x, y)$$

for some $C \geq 1$ and every $x, y \in X$. Of course, this implies that ϕ is an isometry on X when $C = 1$. Otherwise, note that ϕ^{-1} is also bilipschitz with the same constant C , and that ϕ^n is bilipschitz with constant $C^{|n|}$ for each $n \in \mathbf{Z}$. If ϕ^n is actually bilipschitz with a constant that does not depend on n for each $n \in \mathbf{Z}$, then ϕ is an isometry with respect to the metric $\tilde{d}(x, y)$ on X defined in the previous section, and $\tilde{d}(x, y)$ is bounded by a constant multiple of $d(x, y)$.

As in the previous section,

$$(5.2) \quad \rho((x, r), (y, t)) = \max(d(x, y), |r - t|)$$

defines a metric on $X \times \mathbf{R}$ for which the corresponding topology is the product topology. Let Φ be the mapping on $X \times \mathbf{R}$ defined in Section 1, so that

$$(5.3) \quad \begin{aligned} \rho(\Phi(x, r), \Phi(y, t)) &= \rho((\phi(x), r + 1), (\phi(y), t + 1)) \\ &= \max(d(\phi(x), \phi(y)), |r - t|) \end{aligned}$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$. Using this, it is easy to see that Φ is a bilipschitz mapping on $X \times \mathbf{R}$ with constant C with respect to $\rho(\cdot, \cdot)$. As before, we would like to define a distance function

$$(5.4) \quad \delta(q_2((x, r)), q_2((y, t)))$$

on Y_2 that looks locally like (5.2), at least when $|r|$ and $|t|$ are not too large.

Let $x, y \in X$ and $r, t \in \mathbf{R}$ be given, and suppose that $x', y' \in X$ and $r', t' \in \mathbf{R}$ satisfy

$$(5.5) \quad q_2((x, r)) = q_2((x', r')), \quad q_2((y, t)) = q_2((y', t')).$$

This implies that

$$(5.6) \quad r \equiv r' \text{ and } t \equiv t' \text{ modulo } \mathbf{Z},$$

and hence

$$(5.7) \quad r - t \equiv r' - t' \text{ modulo } \mathbf{Z}.$$

Note that x', y' are uniquely determined by (5.5) and r', t' . If we restrict our attention to r', t' in a bounded set, then there are only finitely many possibilities for them, and thus for x', y' .

We can always choose $r', t' \in \mathbf{R}$ so that (5.5) holds and

$$(5.8) \quad |r' - t'| \leq 1/2,$$

by adding suitable integers to r' or t' . One can also get

$$(5.9) \quad \left| \frac{r' + t'}{2} \right| \leq \frac{1}{2},$$

by adding a suitable integer to both r' and t' , which does not affect (5.8). Under these conditions,

$$(5.10) \quad |r'|, |t'| \leq 3/4,$$

because the distance from r' or t' to $(r' + t')/2$ is equal to $|r' - t'|/2$.

Put

$$(5.11) \quad \begin{aligned} & \delta(q_2((x, r)), q_2((y, t))) \\ &= \min\{\rho((x', r'), (y', t')) : x', y' \in X, r', t' \in \mathbf{R} \\ & \quad \text{satisfy (5.5), (5.8), and (5.10)}\}. \end{aligned}$$

As in the previous paragraphs, every pair of points in Y_2 can be represented in this way, and there are only finitely many such representations. Thus the minimum in (5.11) makes sense, and is a nonnegative real number. If

$$(5.12) \quad q_2((x, r)) = q_2((y, t)),$$

then we can choose $x', y' \in X$ and $r', t' \in \mathbf{R}$ such that $|r'| \leq 1/2 \leq 1$ and $r' = t'$, which implies that (5.11) is equal to 0. Otherwise, if $q_2((x, r)) \neq q_2((y, t))$, then (5.11) is the minimum of finitely many positive real numbers, and hence is positive too. Clearly (5.11) is symmetric in $q_2((x, r))$ and $q_2((y, t))$. However,

(5.11) does not normally satisfy the triangle inequality, and we shall come back to that soon.

Suppose that

$$(5.13) \quad |r|, |t| \leq 3/4 \quad \text{and} \quad |r - t| \leq 1/2,$$

so that

$$(5.14) \quad \delta(q_2((x, r)), q_2((y, t))) \leq \rho((x, r), (y, t)),$$

because x , y , r , and t are admissible competitors for the minimum in (5.11). If

$$(5.15) \quad |r - t| < 1/2,$$

then we also have that

$$(5.16) \quad \delta(q_2((x, r)), q_2((y, t))) \geq C^{-1} \rho((x, r), (y, t)),$$

where C is as in (5.1). To see this, suppose that $x', y' \in X$ and $r', t' \in \mathbf{R}$ satisfy (5.5), (5.8), and (5.10), and that $r' \neq r$ or $t' \neq t$. Observe that

$$(5.17) \quad r' - t' = r - t$$

in this situation, because of (5.7), (5.8), and (5.15). Moreover,

$$(5.18) \quad |r' - r|, |t' - t| \leq 3/2,$$

by (5.10) and (5.13), which implies that

$$(5.19) \quad |r' - r|, |t' - t| \leq 1,$$

because of (5.6). Thus

$$(5.20) \quad r' - r = t' - t = 1 \text{ or } -1$$

under these conditions, by (5.6), (5.17), (5.19), and the hypothesis that $r' \neq r$ or $t' \neq t$. This implies that either $x' = \phi(x)$ and $y' = \phi(y)$, or $x' = \phi^{-1}(x)$ and $y' = \phi^{-1}(y)$, because of (5.5). In both cases, we get that

$$(5.21) \quad d(x', y') \geq C^{-1} d(x, y),$$

by (5.1). It follows that

$$(5.22) \quad \rho((x', r'), (y', t')) \geq C^{-1} \rho((x, r), (y, t)),$$

using also (5.17). This shows that (5.16) holds when $|r - t| < 1/2$, as desired.

By construction,

$$(5.23) \quad \delta(q_2((x, r)), q_2((y, t))) \geq \text{dist}(r - t, \mathbf{Z})$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$, and hence

$$(5.24) \quad \delta(q_2((x, r)), q_2((y, t))) \geq 1/2$$

when $|r - t| = 1/2$. Combining this with (5.16), we get that

$$(5.25) \quad \delta(q_2((x, r)), q_2((y, t))) \geq \min(C^{-1} \rho((x, r), (y, t)), 1/2)$$

when r, t satisfy (5.13).

As mentioned earlier, $\delta(\cdot, \cdot)$ does not normally satisfy the triangle inequality. To fix this, let $q_2((x, r))$ and $q_2((y, t))$ be any two elements of Y_2 , and consider all finite sequences

$$(5.26) \quad q_2((x_1, r_1)), \dots, q_2((x_{n+1}, r_{n+1}))$$

of elements of Y_2 connecting $q_2((x, r))$ to $q_2((y, t))$, in the sense that

$$(5.27) \quad q_2((x_1, r_1)) = q_2((x, r)) \quad \text{and} \quad q_2((x_{n+1}, r_{n+1})) = q_2((y, t)).$$

Put

$$(5.28) \quad \begin{aligned} & \delta_0(q_2((x, r)), q_2((y, t))) \\ &= \inf \left\{ \sum_{j=1}^n \delta(q_2((x_j, r_j)), q_2((x_{j+1}, r_{j+1}))) : \right. \\ & \left. x_1, \dots, x_{n+1} \in X, r_1, \dots, r_{n+1} \in \mathbf{R} \text{ satisfy (5.27)} \right\}, \end{aligned}$$

so that the infimum is taken over all finite sequences of elements of Y_2 connecting $q_2((x, r))$ to $q_2((y, t))$. In particular,

$$(5.29) \quad \delta_0(q_2((x, r)), q_2((y, t))) \leq \delta(q_2((x, r)), q_2((y, t))),$$

since one can take $n = 2$, $x_1 = x$, $r_1 = r$, $x_2 = y$, and $r_2 = t$. Of course, (5.28) is nonnegative and symmetric in $q_2((x, r))$ and $q_2((y, t))$, because of the corresponding properties of $\delta(\cdot, \cdot)$. By construction, $\delta_0(\cdot, \cdot)$ satisfies the triangle inequality

$$(5.30) \quad \begin{aligned} & \delta_0(q_2((x, r)), q_2((z, u))) \\ & \leq \delta_0(q_2((x, r)), q_2((y, t))) + \delta_0(q_2((y, t)), q_2((z, u))) \end{aligned}$$

for every $x, y, z \in X$ and $r, t, u \in \mathbf{R}$. This is basically because any finite sequences of elements of Y_2 connecting $q_2((x, r))$ to $q_2((y, t))$ and connecting $q_2((y, t))$ to $q_2((z, u))$ can be combined to get a finite sequence of elements of Y_2 connecting $q_2((x, r))$ to $q_2((z, u))$.

Let $q_2((x, r))$, $q_2((y, t))$ be any two elements of Y_2 again, and let (5.26) be a finite sequence of elements of Y_2 that satisfies (5.27). Observe that

$$(5.31) \quad \sum_{j=1}^n \delta(q_2((x_j, r_j)), q_2((x_{j+1}, r_{j+1}))) \geq \sum_{j=1}^n \text{dist}(r_j - r_{j+1}, \mathbf{Z}),$$

by (5.23). The triangle inequality (4.25) for $\text{dist}(a, \mathbf{Z})$ implies that

$$(5.32) \quad \sum_{j=1}^n \text{dist}(r_j - r_{j+1}, \mathbf{Z}) \geq \text{dist}(r_1 - r_{n+1}, \mathbf{Z}) = \text{dist}(r - t, \mathbf{Z}),$$

using the fact that $r_1 \equiv r$ and $r_{n+1} \equiv t$ modulo \mathbf{Z} in the second step, which follows from (5.27). Thus

$$(5.33) \quad \sum_{j=1}^n \delta(q_2((x_j, r_j)), q_2((x_{j+1}, r_{j+1}))) \geq \text{dist}(r - t, \mathbf{Z}),$$

and hence

$$(5.34) \quad \delta_0(q_2((x, r)), q_2((y, t))) \geq \text{dist}(r - t, \mathbf{Z}),$$

by taking the infimum of the sums on the left side of (5.33).

As before, any two elements of Y_2 can be represented as $q_2((x, r))$, $q_2((y, t))$ for some $x, y \in X$ and $r, t \in \mathbf{R}$ such that

$$(5.35) \quad |r - t| \leq \frac{1}{2} \quad \text{and} \quad \left| \frac{r + t}{2} \right| \leq \frac{1}{2},$$

and hence $|r|, |t| \leq 3/4$. Let (5.26) be a finite sequence of elements of Y_2 such that (5.27) again. We may as well choose $x_1, \dots, x_{n+1} \in X$ and $r_1, \dots, r_{n+1} \in \mathbf{R}$ such that $x_1 = x$, $r_1 = r$, and

$$(5.36) \quad |r_j - r_{j+1}| = \text{dist}(r_j - r_{j+1}, \mathbf{Z})$$

for $j = 1, \dots, n$. Suppose for the moment that

$$(5.37) \quad \sum_{j=1}^n \text{dist}(r_j - r_{j+1}, \mathbf{Z}) < 1/2,$$

so that

$$(5.38) \quad \sum_{j=1}^n |r_j - r_{j+1}| < 1/2.$$

In particular,

$$(5.39) \quad |r - r_{n+1}| = |r_1 - r_{n+1}| < 1/2,$$

which implies that

$$(5.40) \quad |r_{n+1} - t| \leq |r_{n+1} - r_1| + |r - t| < 1/2 + 1/2 = 1,$$

by the first part of (5.35). It follows that $r_{n+1} = t$ under these conditions, since $r_{n+1} \equiv t$ modulo \mathbf{Z} , by (5.27). Using (5.37) again, we get that

$$(5.41) \quad |r - r_l| + |r_l - t| < 1/2$$

for each $l = 1, \dots, n + 1$, and hence

$$(5.42) \quad \left| r_l - \frac{r + t}{2} \right| \leq \frac{|r_l - r| + |r_l - t|}{2} < \frac{1}{4}.$$

Thus $|r_l| < 3/4$ for each $l = 1, \dots, n+1$, and of course $|r_j - r_{j+1}| < 1/2$ for each $j = 1, \dots, n$, by (5.38). This permits us to apply (5.16) to $q_2((x_j, r_j))$ and $q_2((x_{j+1}, r_{j+1}))$ for each $j = 1, \dots, n$, to get that

$$(5.43) \quad \delta(q_2((x_j, r_j)), q_2((x_{j+1}, r_{j+1}))) \geq C^{-1} \rho((x_j, r_j), (x_{j+1}, r_{j+1}))$$

for each $j = 1, \dots, n$. Because $\rho(\cdot, \cdot)$ is a metric on $X \times \mathbf{R}$, and hence satisfies the triangle inequality, we get that

$$(5.44) \quad \begin{aligned} \sum_{j=1}^n \delta(q_2((x_j, r_j)), q_2((x_{j+1}, r_{j+1}))) &\geq C^{-1} \sum_{j=1}^n \rho((x_j, r_j), (x_{j+1}, r_{j+1})) \\ &\geq C^{-1} \rho((x, r), (y, t)). \end{aligned}$$

Otherwise, if (5.37) does not hold, then

$$(5.45) \quad \sum_{j=1}^n \delta(q_2((x_j, r_j)), q_2((x_{j+1}, r_{j+1}))) \geq 1/2,$$

by (5.31). Combining this with the previous case, and taking the infimum of the sums on the left side, we get that

$$(5.46) \quad \delta_0(q_2((x, r)), q_2((y, t))) \geq \min(C^{-1} \rho((x, r), (y, t)), 1/2)$$

when r and t satisfy (5.35). Note that we also have

$$(5.47) \quad \delta_0(q_2((x, r)), q_2((y, t))) \leq \rho((x, r), (y, t))$$

under these conditions, by (5.14) and (5.29).

Suppose now that X is bounded with diameter less than or equal to k for some $k \geq 1/2$, so that

$$(5.48) \quad d(x, y) \leq k$$

for every $x, y \in X$. Of course, this can always be arranged by replacing $d(x, y)$ with the minimum of $d(x, y)$ and k , as in the previous section. Alternatively, if X is already bounded with respect to $d(x, y)$, then one can get this condition by multiplying $d(x, y)$ by a suitable positive constant. In both cases, one can check that the bilipschitz condition for ϕ would be maintained.

Using (5.48) and the definition (5.2) of $\rho(\cdot, \cdot)$, we get that

$$(5.49) \quad \rho((x, r), (y, t)) \leq k$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$ such that $|r - t| \leq 1/2$. This implies that

$$(5.50) \quad \delta_0(q_2((x, r)), q_2((y, t))) \leq \delta(q_2((x, r)), q_2((y, t))) \leq k,$$

for every $x, y \in X$ and $r, t \in \mathbf{R}$, because of (5.29) and the definition (5.11) of $\delta(q_2((x, r)), q_2((y, t)))$. If r, t satisfy (5.35), then we get that

$$(5.51) \quad \rho((x, r), (y, t)) \leq \max(C, 2k) \delta_0(q_2((x, r)), q_2((y, t))),$$

by combining (5.46) and (5.49).

6 Nonnegative Borel measures

Let us return to the setting of Section 1. If μ is a nonnegative Borel measure on X , then we would like to have a corresponding product measure on $X \times \mathbf{R}$, using Lebesgue measure on \mathbf{R} . Of course, the standard product measure construction applies when μ is at least σ -finite on X . It is better for X to also have a countable base for its topology, so that there is a countable base for the topology of $X \times \mathbf{R}$ consisting of products of open subsets of X and \mathbf{R} . This implies that open subsets of $X \times \mathbf{R}$ are measurable with respect to the usual product σ -algebra, and hence that Borel sets in $X \times \mathbf{R}$ are measurable too. Alternatively, if X is a locally compact Hausdorff space, then one might start with a Borel measure μ on X with suitable regularity properties, and look for a product Borel measure on $X \times \mathbf{R}$ with similar regularity properties. More precisely, one can view this in terms of nonnegative linear functionals on spaces of continuous functions with compact support, using the Riesz representation theorem.

At any rate, such a product measure on $X \times \mathbf{R}$ is invariant under translations on \mathbf{R} , because Lebesgue measure is invariant under translations. If μ is invariant under ϕ on X , then the product measure is invariant under Φ on $X \times \mathbf{R}$. One can then localize to get a measure on Y_2 that is invariant under the mappings A_r corresponding to translation on \mathbf{R} . If μ is not invariant under ϕ , then one can still get measures on Y_2 , by restricting the product measure to the product of X with an interval I in \mathbf{R} with length 1. Of course, the resulting measures on Y_2 will depend on I , but under suitable conditions on ϕ and μ , they may be reasonably similar.

Suppose that X is compact, and that the topology on X is determined by a metric $d(x, y)$. If ϕ is bilipschitz with respect to this metric, then one can get a metric on Y_2 that looks locally approximately like a product metric on $X \times \mathbf{R}$, as in the previous section. If μ is Ahlfors regular of some dimension t , then one can get an Ahlfors regular measure on Y_2 of dimension $t + 1$, even if μ is not invariant under ϕ . More precisely, if μ is Ahlfors regular on X of dimension t , then μ is approximately the same as t -dimensional Hausdorff measure on X , in the sense that each is bounded by constant multiples of the other. Hausdorff measure in any dimension is approximately preserved to within bounded factors by a bilipschitz mapping, which implies that μ is approximately preserved by ϕ to within bounded factors under these conditions.

Even if μ is not Ahlfors regular, it may be approximately preserved to within bounded factors by ϕ , so that one can get corresponding measures on Y_2 that are at least comparable to each other. If μ is a doubling measure on X , and if ϕ is bilipschitz or at least quasisymmetric on X , then μ is transformed by ϕ to a doubling measure on X , but the new measure may not be comparable to μ . If μ is a doubling measure on X which is approximately preserved to within bounded factors by ϕ , and if ϕ is bilipschitz, then one can get doubling measures on Y_2 from μ , as before. Although quasisymmetry of ϕ on X is a very natural geometric condition, it is not by itself so convenient for looking at geometric structures on $X \times \mathbf{R}$, and hence Y_2 . However, if ϕ is a quasisymmetric mapping on X that approximately preserves a nontrivial doubling measure μ

on X to within bounded factors, then one can use that to get another geometric structure on X that is approximately preserved by ϕ to within bounded factors, at least if X is also uniformly perfect.

7 Some examples

Let B be a finite set with at least two elements, and let X be the set of doubly-infinite sequences $x = \{x_j\}_{j=-\infty}^{\infty}$ such that $x_j \in B$ for each $j \in \mathbf{Z}$. Equivalently, X is the Cartesian product of a family of copies of B , indexed by \mathbf{Z} . Thus X is a compact Hausdorff topological space, with respect to the product topology associated to the discrete topology on each copy of B . Let ϕ be the shift mapping defined by

$$(7.1) \quad \phi(x) = \{x_{j-1}\}_{j=-\infty}^{\infty},$$

which is a homeomorphism from X onto itself. Also let Φ , Y_2 , etc., be as in Section 1, using this X and ϕ .

Suppose that E_0 is a nonempty open set in X , and let x be an element of E_0 . Because of the way that the product topology is defined on X , there is a nonnegative integer n such that E_0 contains every $y \in X$ that satisfies

$$(7.2) \quad y_j = x_j \text{ for each } j \in \mathbf{Z} \text{ with } |j| \leq n.$$

If $\phi(E_0) = E_0$, then $\phi^k(E_0) = E_0$ for every $k \in \mathbf{Z}$, and hence E_0 also contains every $z \in X$ such that $y = \phi^k(z)$ satisfies (7.2) for some $k \in \mathbf{Z}$. If E_0 is a closed set in X too, then it follows that $E_0 = X$, again because of the way that the product topology is defined on X . This shows that $E_0 = X$ when E_0 is a nonempty open and closed subset of X that is invariant under ϕ , which implies that Y_2 is connected, as in Section 2.

Let w be a nonnegative real-valued function on B such that

$$(7.3) \quad \sum_{b \in B} w(b) = 1.$$

Thus w defines a probability measure on B in the obvious way, and we let $\mu = \mu_w$ be the corresponding product measure on X , using the probability measure on B associated to w on each factor. One can first define the corresponding integral of a continuous real-valued function on X as a limit of suitable finite sums, and then get μ_w as a regular Borel measure on X from the Riesz representation theorem. Of course, μ_w is invariant under ϕ for every w , since μ_w is defined using the same probability measure on each copy of B in the product. Note that there is a countable base for the topology of X , by standard arguments.

Let $x, y \in X$ be given, with $x \neq y$, and let $n(x, y)$ be the largest nonnegative integer such that

$$(7.4) \quad x_j = y_j \text{ for every } j \in \mathbf{Z} \text{ with } -n+1 \leq j \leq n,$$

which holds trivially when $n = 0$. If $x = y$, then one can put $n(x, y) = +\infty$. It is easy to see that

$$(7.5) \quad n(x, y) = n(y, x)$$

for every $x, y \in X$, and that

$$(7.6) \quad n(x, z) \geq \min(n(x, y), n(y, z))$$

for every $x, y, z \in X$. Observe also that

$$(7.7) \quad n(x, y) - 1 \leq n(\phi(x), \phi(y)) \leq n(x, y) + 1$$

for every $x, y \in X$.

Let a be a positive real number which is strictly less than 1, and put

$$(7.8) \quad d_a(x, y) = a^{n(x, y)}$$

for every $x, y \in X$, which is interpreted as being equal to 0 when $x = y$. Thus $d_a(x, y) > 0$ when $x \neq y$,

$$(7.9) \quad d_a(x, y) = d_a(y, x)$$

for every $x, y \in X$, and

$$(7.10) \quad d_a(x, z) \leq \max(d_a(x, y), d_a(y, z))$$

for every $x, y, z \in X$, by (7.5) and (7.6). This shows that $d_a(x, y)$ is a metric on X for each $a \in (0, 1)$, and in fact $d_a(x, y)$ is an ultrametric on X , since it satisfies the ultrametric version (7.10) of the triangle inequality. By construction, the topology on X determined by $d_a(x, y)$ is the same as the product topology described earlier for each $a \in (0, 1)$. In particular, these ultrametrics on X are topologically equivalent, and indeed we have that

$$(7.11) \quad d_a(x, y)^\alpha = d_{a^\alpha}(x, y)$$

for every $a \in (0, 1)$, $\alpha > 0$, and $x, y \in X$.

It follows from (7.7) that

$$(7.12) \quad a d_a(x, y) \leq d_a(\phi(x), \phi(y)) \leq (1/a) d_a(x, y)$$

for every $a \in (0, 1)$ and $x, y \in X$, so that ϕ is bilipschitz with constant $C = 1/a$ with respect to $d_a(x, y)$. However, one can also check that the collection of iterates ϕ^k of ϕ with $k \in \mathbf{Z}$ is not equicontinuous at any point in X with respect to $d_a(x, y)$ for any $a \in (0, 1)$. If $d(x, y)$ is any metric on X that determines the same topology on X , then the identity mapping on X is uniformly continuous as a mapping from X equipped with $d(x, y)$ into X equipped with $d_a(x, y)$ for any $a \in (0, 1)$, because X is compact. This implies that the collection of iterates ϕ^k of ϕ with $k \in \mathbf{Z}$ is not equicontinuous with respect to any metric $d(x, y)$ on X that determines the same topology on X .

By construction,

$$(7.13) \quad d_a(x, y) \leq 1$$

for each $x, y \in X$ and $a \in (0, 1)$, and equality holds when $x_0 \neq y_0$. The closed ball in X with respect to $d_a(x, y)$ centered at some point $x \in X$ and with radius

a^n for some nonnegative integer n is the same as the set of $y \in X$ that satisfy (7.4). If w is a positive real-valued function on B that satisfies (7.3), then one can check that the corresponding probability measure μ_w on X is a doubling measure with respect to $d_a(x, y)$.

Suppose now that w corresponds to the uniform distribution on B , so that

$$(7.14) \quad w(b) = 1/\#B,$$

where $\#B$ denotes the number of elements of B . In this case, the measure with respect to μ_w of a closed ball in X with respect to $d_a(x, y)$ with radius a^n for some nonnegative integer n is

$$(7.15) \quad (\#B)^{-2n}.$$

If we put $t = -2 \log(\#B)/\log a$, then $t > 0$ and

$$(7.16) \quad (a^n)^t = (\#B)^{-2n}$$

for each $n \geq 0$, which implies that μ_w is Ahlfors regular on X with respect to $d_a(x, y)$, with dimension t . In particular, the Hausdorff dimension of X with respect to $d_a(x, y)$ is equal to t .

8 Snowflake metrics

If $d(x, y)$ is a metric on a set X , then $d(x, y)^\alpha$ is also a metric on X when $0 < \alpha < 1$, which determines the same topology on X . However, $d(x, y)^\alpha$ does not necessarily satisfy the triangle inequality when $\alpha > 1$, even when X is the unit interval with the standard metric. It is easy to see that $d(x, y)^\alpha$ is still a quasi-metric on X when $\alpha > 1$, which means that it satisfies a weaker version of the triangle inequality with an extra constant factor on the right side, and which is adequate in many situations. Of course, if $d(x, y)$ is an ultrametric on X , then $d(x, y)^\alpha$ is an ultrametric on X for each $\alpha > 0$. Note that the Hausdorff dimension of X with respect to $d(x, y)^\alpha$ is equal to the Hausdorff dimension of X with respect to $d(x, y)$ divided by α .

If ϕ is a bilipschitz mapping from X onto itself with respect to $d(x, y)$ with constant C , then ϕ is also bilipschitz with respect to $d(x, y)^\alpha$, with constant C^α . In particular, if ϕ is an isometry with respect to $d(x, y)$, then ϕ preserves $d(x, y)^\alpha$ for each α . Thus one can repeat the same types of constructions as before, with $d(x, y)$ replaced with $d(x, y)^\alpha$. This was already built in the examples discussed in the previous section, using the parameter $a \in (0, 1)$. If $X = [0, 1]$ with the standard metric $d(x, y) = |x - y|$ and $0 < \alpha < 1$, then X is basically a snowflake curve of dimension $1/\alpha$ with respect to $d(x, y)^\alpha$.

If $d(x, y)$ is any metric on a set X and $0 < \alpha < 1$, then one can check that every continuous curve in X with finite length with respect to $d(x, y)^\alpha$ is constant. Consider the metric on $X \times \mathbf{R}$ defined by

$$(8.1) \quad \rho_\alpha((x, r), (y, t)) = \max(d(x, y)^\alpha, |r - t|),$$

which is the analogue of (4.2), (5.2) with $d(x, y)$ replaced by $d(x, y)^\alpha$. If γ is any continuous curve in $X \times \mathbf{R}$ with finite length with respect to (8.1), then the projection of γ into X has finite length with respect to $d(x, y)^\alpha$, and hence is constant. This means that γ is contained in a line $\{x\} \times \mathbf{R}$ for some $x \in X$, and hence corresponds to a curve of finite length in the real line, with the standard metric. Similarly, if Y_2 is equipped with a metric that looks locally like (8.1), as before, then any continuous curve in Y_2 with finite length has to be contained in the image of a line $\{x\} \times \mathbf{R}$ under the usual quotient mapping q_2 .

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